Dualized Type Theory

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Long-term Goal

Inductive Data:

Coinductive Data:

Mixed Ind-Coind Data:

- Coq is not type safe [Giménez:1997].
Bi-intuitionistic (BINT) Logic

• Intuitionistic logic with perfect duality.
• The dual of implication is subtraction or exclusion.
• First studied by the Cycilia Rauszer in the 70’s.
BINT Logic and Type Theory

- **Symmetric Comb. Logic**: Filinski.
- **Subtractive logic**: Crolard.
- **Logic for Pragmatics**: Bellin, and Biasi and Aschieri.
- **Nested Sequents**: Rajeev Goré, Linda Postniece & Alwen Tiu.
- **Labeled BINT**: Pinto and Uustalu.

We really want to stick to sequent calculi.
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- **Labled BINT**: Pinto and Uustalu.

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Labeled BINT

LK+Subtraction:

\[ A_1, \ldots, A_j \vdash B_1, \ldots, B_k \]

Labeled BINT:

\[ n_1 : A_1, \ldots, n_j : A_j \vdash_G m_1 : B_1, \ldots, m_k : B_k \]
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LK+Subtraction:

\[ A_1, \ldots, A_j \vdash B_1, \ldots, B_k \]

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Labeled BINT

\[
\begin{align*}
    n' & \not\in |G|, |\Gamma|, |\Delta| \\
    \Gamma, n' : T_1 & \vdash_{G \cup \{(n,n')\}} n' : T_2, \Delta \\
    \Gamma & \vdash_G n : T_1 \supset T_2, \Delta
\end{align*}
\]

\[\text{IMPR}\]

\[
\begin{align*}
    n' G n \\
    \Gamma & \vdash_G n' : T_1, \Delta \\
    \Gamma, n' : T_2 & \vdash_G \Delta
\end{align*}
\]

\[\text{SUBR}\]

\[
\begin{align*}
    \Gamma & \vdash_G n : T_1 \prec T_2, \Delta
\end{align*}
\]
Labeled BINT

\[
\begin{align*}
\Gamma \vdash G \cup \{(n,n)\} & \quad \Delta \\
\hline
\Gamma \vdash G \quad \Delta & \quad \text{REFL}
\end{align*}
\]

\[
\begin{align*}
n_1Gn_2 \\
n_2Gn_3
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash G \cup \{(n_1,n_3)\} & \quad \Delta \\
\hline
\Gamma \vdash G \quad \Delta & \quad \text{TRANS}
\end{align*}
\]

\[
\begin{align*}
nGn' \\
\Gamma, n : T, n' : T \vdash G \quad \Delta
\end{align*}
\]

\[
\begin{align*}
\Gamma, n : T \vdash G \quad \Delta & \quad \text{MONO_L}
\end{align*}
\]

\[
\begin{align*}
n'Gn \\
\Gamma \vdash G \quad \Delta
\end{align*}
\]

\[
\begin{align*}
n' \vdash T, \Delta & \quad \text{MONO_R}
\end{align*}
\]

\[
\begin{align*}
n \vdash T, \Delta
\end{align*}
\]
Dualized Intuitionistic Logic (DIL)

- A simplification of labeled BINT.
- A new dualized syntax.
- Pushed the refl and trans rules to the leaves.
- Removed the monotonicity rules.
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Dualized Intuitionistic Logic (DIL)

\[ G \vdash n \preceq^*_p n' \]
\[ G; \Gamma, pA \at n, \Gamma' \vdash pA \at n' \quad \text{AX} \]
\[ G; \Gamma \vdash \langle p \rangle \at n \quad \text{UNIT} \]

\[ G; \Gamma \vdash pA \at n \quad G; \Gamma \vdash pB \at n \]
\[ G; \Gamma \vdash p(A \land_p B) \at n \quad \text{AND} \]

\[ G; \Gamma \vdash pA_d \at n \]
\[ G; \Gamma \vdash p(A_1 \land_{\overline{p}_2} A_2) \at n \quad \text{ANDBAR} \]

\[ n' \not\in |G|, |\Gamma| \]
\[ (G, n \preceq^p n'); \Gamma, pA \at n' \vdash pB \at n' \]
\[ G; \Gamma \vdash p(A \rightarrow_p B) \at n \quad \text{IMP} \]

\[ G \vdash n \preceq^*_p n' \]
\[ G; \Gamma \vdash \overline{p}A \at n' \quad G; \Gamma \vdash pB \at n' \]
\[ G; \Gamma \vdash p(A \rightarrow_{\overline{p}} B) \at n \quad \text{IMPBAR} \]

\[ G; \Gamma, \overline{p}A \at n \vdash +B \at n' \quad G; \Gamma, \overline{p}A \at n \vdash -B \at n' \]
\[ G; \Gamma \vdash pA \at n \quad \text{CUT} \]
Dualized Intuitionistic Logic (DIL)

Labeled BINT:
\[ n_1 : A_1, \ldots, n_j : A_j \vdash_G m_1 : B_1, \ldots, m_k : B_k \]

DIL:
\[ G; + A_1 @ n_1, \ldots, + A_j @ n_j, - B_2 @ m_2, \ldots, - B_k @ m_k \vdash + B_1 @ m_1 \]
\[ G; p_1 A_1 @ n_1, \ldots, p_2 A @ n_2 \vdash p B @ n \]
Dualized Intuitionistic Logic (DIL)

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\[ G; p_1 A_1 @ n_1, \ldots, p_2 A @ n_2 \vdash p B @ n \]
Dualized Intuitionistic Logic (DIL)

\[
\begin{align*}
& n' \not\in |G|, |\Gamma| \\
& (G, n \leq_p n'); \Gamma, p \ A @ n' \vdash p \ B @ n' \\
& G; \Gamma \vdash p (A \rightarrow_p B) @ n
\end{align*}
\]

\text{IMP}

\[
\begin{align*}
& G \vdash n \leq_{\bar{p}} n' \\
& G; \Gamma \vdash \bar{p} \ A @ n' \\
& G; \Gamma \vdash p \ B @ n' \\
& G; \Gamma \vdash p (A \rightarrow_{\bar{p}} B) @ n
\end{align*}
\]

\text{IMP\bar{B}AR}
Consistency and Completeness

- Consistency was proven (in Agda) w.r.t. the following notion of validity*:

\[
[G; \Gamma \vdash p \ A @ n]_N = \text{if } [G]_N \text{ and } [\Gamma]_N, \text{ then } p[A]_{(N \ n)}
\]

- Completeness is shown by reduction to L:

If \( \downarrow G \vdash \Gamma' \vdash + A @ n \) is an activation of the derivable L-sequent \( \Gamma \vdash_G \Delta \), then \( \downarrow G \vdash \Gamma' \vdash + A @ n \) is derivable.

* https://github.com/heades/DIL-consistency
Consistency and Completeness

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\]

• Completeness is shown by reduction to L:

If \( \overline{\vdash} G \vdash \Delta \); \( \Gamma' \vdash + A @ n \) is an activation of the derivable L-sequent \( \Gamma \vdash_G \Delta \), then \( \overline{\vdash} G \vdash \Delta \); \( \Gamma' \vdash + A @ n \) is derivable.

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Consistency and Completeness

- Consistency was proven (in Agda) w.r.t. the following notion of validity:

\[
\left\{G; \Gamma, \bar{p} A @ n \vdash p A @ n \right\} \iff \left\{G; \Gamma, \bar{p} A @ n \vdash + B @ n' \right\} \text{ and } \left\{G; \Gamma, \bar{p} A @ n \vdash - B @ n' \right\}
\]

- Completeness is shown by reduction to L:

\[
\text{If } p B @ n' \in (\Gamma, \bar{p} A @ n), \text{ then } p B @ n' \in (\Gamma, \bar{p} A @ n) \text{ is an activation of the derivable L-sequent } G; \Gamma, \bar{p} A @ n \vdash p B @ n'.
\]

\[
G; \Gamma, \bar{p} A @ n \vdash p A @ n \implies G; \Gamma \vdash p A @ n
\]

\[
\text{CUT} \quad \text{AX Cut}
\]

\[
\text{axCutBar}
\]
Dualized Type Theory

\[
\begin{align*}
G \vdash n \leq^*_{p} n' & \quad \text{Ax} \\
G; \Gamma, x : p A \vdash n, \Gamma' \vdash x : p A \vdash n' & \quad \text{UNIT} \\
G; \Gamma \vdash t_1 : p A \vdash n & \quad \text{AND} \\
G; \Gamma \vdash t_2 : p B \vdash n & \quad \text{ANDBAR} \\
G; \Gamma \vdash (t_1, t_2) : p (A \land p B) \vdash n & \\
G; \Gamma \vdash t : p A_d \vdash n & \\
G; \Gamma \vdash \text{in}_d t : p (A_1 \land \lnot p A_2) \vdash n & \\
n' \not\in |G|, |\Gamma| & \\
(G, n \leq^p n'); \Gamma, x : p A \vdash n' \vdash t : p B \vdash n' & \quad \text{IMP} \\
G; \Gamma \vdash \lambda x. t : p (A \rightarrow_{p} B) \vdash n & \\
G \vdash n \leq^*_{\lnot p} n' & \\
G; \Gamma \vdash t_1 : \lnot p A \vdash n' & \quad \text{IMPBAR} \\
G; \Gamma \vdash t_2 : p B \vdash n' & \\
G; \Gamma \vdash \langle t_1, t_2 \rangle : p (A \rightarrow_\lnot p B) \vdash n & \\
G; \Gamma, x : \lnot p A \vdash n \vdash t_1 : + B \vdash n' & \\
G; \Gamma, x : \lnot p A \vdash n \vdash t_2 : - B \vdash n' & \quad \text{CUT} \\
G; \Gamma \vdash \nu x.t_1 \cdot t_2 : p A \vdash n & \\
\end{align*}
\]
Dualized Type Theory

\[ \Gamma' \overset{\text{def}}{=} \Gamma, \ y : - B @ n \]

\[ G; \Gamma \vdash \lambda x. t : + (A \to_+ B) @ n \quad \frac{G; \Gamma \vdash t' : + A @ n \quad G; \Gamma' \vdash y : - B @ n}{G; \Gamma' \vdash \langle t', y \rangle : - (A \to_+ B) @ n} \quad \text{AX} \]

\[ G; \Gamma' \vdash \langle t', y \rangle : - (A \to_+ B) @ n \quad \frac{G; \Gamma' \vdash \langle t', y \rangle : - (A \to_+ B) @ n}{G; \Gamma \vdash \nu y. \lambda x. t \cdot \langle t', y \rangle : + B @ n} \quad \text{IMPBAR} \]

\[ G; \Gamma \vdash \nu y. \lambda x. t \cdot \langle t', y \rangle : + B @ n \]
Dualized Type Theory

- Metatheory of DTT:
  - Type preservation.
  - Strong normalization.
  - By forgetting the labels and proving SN of LK+subtraction using classical realizability.
Plans for the Future

• Add inductive and coinductive types.
• Need a categorical model of DTT.
• Preordered Categories:
Plans for the Future

Category: $(\mathcal{P}, \mathcal{C})$

Objects: $A@n_1, B@n_2, \cdots$

Morphisms: $A_1@n_1, A_2@n_2, \ldots, A_i@n_i \xrightarrow{f^M} B@n$

DIL-sequent:

$G; +A_1@n_1, \ldots, +A_2@n_2 \vdash +B@n$

$[A_1]@[n_1], \ldots, [A_2]@[n_2] \xrightarrow{f[G]} [B]@[n]$
Thank you!